Comparative Study Of Numerical Method –Spline, Finite Difference And Finite Element For Solving One Dimensional Hyperbolic Partial Differential Equation

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Abstract:

The present work describes spline collocation method for solving flow of electricity cable in transmission line. The mathematical problem gives rise to solve a partial differential equation with one space variable which is of hyperbolic type. The method involves the solution of algebraic linear equation which can be written in the matrix form is main advantage. The solution are obtained by spline explicit and spline implicit methods and compared with Finite difference, Finite element and analytic solution to demonstrate the justification and simplicity of the spline approximation.

Key words: Spline collocation, Finite difference, Finite Element Method, Partial differential equation

1 INTRODUCTION:

Two common questions are encountered while the numerical solution to the problem is obtained. The first is about its acceptance whether it is sufficiently close to the true solution or not. If one has an analytic solution then this can be answered very clearly but in either case it is not so easy. One has to be careful while concluding that a particular numerical solution is acceptable when an analytic solution is not available. Normally a method is selected which requires a minimum number of steps, consuming the shortest computational time and yet one that does not produce an excessive errors.

2 SPLINE COLLOCATION METHOD:

For solving linear and nonlinear differential equation with the help of the numerical methods required much computational work and time. Brickley [2] Suggested the method of spline function containing truncated power polynomials to solve a linear boundary value problem Ahlberg et al [1] used cardinal splines for solving differential equations. Doctor et al [4,5] have shown that the method of spline collocation is quite useful for the solution of physical phenomena which give rise to linear parabolic one dimensional partial differential equation. The method demonstrates the use of spline
function. Spline functions are piecewise polynomial and their successive derivatives are continuous. They were used for data interpolation initially. In 1967, Blue [3] suggested the use of spline function for the solution of B.V.P.

\[ y'' = f(x, y, y') \]  \hspace{1cm} (1)

with boundary conditions

\[ G_1[Y(0), Y'(0)] \]

\[ G_2[Y(1), y'(1)] \]  \hspace{1cm} (1a)

The following recurrence relations were used.

\[ S''(x_{i-1}) + 4s''(x_i) + s''(x_{i+1}) = 6h^2(f(x_{i-1}) - 2f(x_i) + f(x_{i+1})) \]  \hspace{1cm} (2)

3. SPLINE FORMULA TO SOLVE HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION WITH ONE SPACE VARIABLES:

The general form of hyperbolic PDE with one space variable x and time variable t is given by

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} ; \quad 0 < x < L, \quad t > 0 \]  \hspace{1cm} (3)

with a Dirichlet boundary conditions, namely

\[ u(0, t) = 0 \]

\[ u(L, t) = 0 \]  \hspace{1cm} (4)

and two initial conditions at t = 0 (Cauchy conditions)

\[ u(x, 0) = f(x) \]

\[ u_t(x, 0) = g(x) \]  \hspace{1cm} (5)

In equation (3), \( c^2 \) is a constant term, it depends upon some physical quantities in case of different problems.

Divide the region \( 0 \leq x \leq L \) into say n sub – intervals each of width \( \Delta x (= h) \) such that \( n\Delta x = L \). The subscript \( j \) denotes time and \( i \) for the positions. The points of subdivisions are
Let $u_{i,j}$ denote the solution of equation (3) at $(i, j)^{th}$ mesh point. Discretize the left hand side of PDE (3) by the central difference formula like finite difference and right side by second derivative of cubic spline $S(x)$ i.e. $S''(x_i)$ at the $(i, j)^{th}$ mesh point, one can get.

\[
(u_{i-1,j+1} + 4u_{i,j+1} + u_{i+1,j+1}) = \frac{2 + 6r^2}{h^2} u_{i-1,j} + \frac{8 - 12r^2}{h^2} u_{i,j} + \frac{2 + 6r^2}{h^2} u_{i+1,j} - [u_{i-1,j-1} + 4u_{i,j-1} + u_{i+1,j-1}]
\]

where \( r = \frac{c\Delta t}{h} \)

Above formula is known as cubic spline explicit formula at to solve hyperbolic PDE of the form (3). It is clear that above formula is applied for all values of $j \geq 1$. However, for $j = 0$, it becomes

\[
u_{i-1,1} + 4u_{i,1} + u_{i+1,1} = \frac{2 + 6r^2}{h^2} u_{i-1,0} + \frac{8 - 12r^2}{h^2} u_{i,0} + \frac{2 + 6r^2}{h^2} u_{i+1,0} - [u_{i-1,-1} + 4u_{i,-1} + u_{i+1,-1}]
\]

where $i = 1(1)n - 1$

The system has a tri-diagonal matrix, which can be solved by any well-known method. After calculating the values of $u$ for $j = 0$, we apply equation (7) for $j \geq 1$, we get $(n - 1)$ simultaneous linear equations in $(n - 1)$ unknowns with tri-diagonal matrix, again $r \leq 1$ is the required condition for convergence and stability of this cubic spline explicit method.

Implicit scheme is unconditionally stable i.e. stable for all the values for $r$. In implicit method the discretization of the differential equation at any mesh point $(i, j)$ is done by replacing time derivative by the central difference formula as done in explicit scheme and the space derivative is replaced by average of second derivatives of cubic spline $S(x)$ at the $(j-1)^{th}$ and $(j+1)^{th}$ level we get

\[
(1 - 3r^2) u_{i+1,j+1} + (4 + 6r^2) u_{i,j+1} + (1 - 3r^2) u_{i-1,j+1} = (3r^2 - 1) u_{i+1,j} - (6r^2 + 4) u_{i,j-1} + (3r^2 - 1) u_{i-1,j-1} + 2(u_{i+1,j} + 4u_{i,j} + u_{i-1,j})
\]

where \( r = \frac{c\Delta t}{h} \); $i = 1(1)n - 1$
The above equation (11) is known as cubic spline implicit formula to solve hyperbolic PDE of the form (3). Like explicit scheme, described as above, the equation (11) gives \((n - 1)\) simultaneous linear equations in \((n - 1)\) unknowns with the coefficient matrix of tri-diagonal form. For \(j = 0\), here we can also use the initial condition in similar manner described as above.

4. THE FLOW OF ELECTRICITY IN THE TRANSMISSION LINES:

This is the study of the derivation of partial differential equations from physical principals, we assume the flow of electricity in a cable of transmission line. We assume the cable to be imperfectly insulated so that there is both capacitance and current leakage to ground. Figure (4.1 a) shows such a cable with the electromotive source \(x=0\) and load \(x=1\).

![Cable Diagram](image)

where \(u\) stands for either \(i(x, t)\). \(L\) is the inductance and \(C\) is the capacitance per unit length of the transmission line i.e. cable. This equation is known as hyperbolic PDE with one space variable \(x\) and time variable \(t\).

Let \(l = \text{length of the transmission line} = 1\) and using following initial and boundary conditions, the solution of above equation (12) i.e. the current distribution is obtained as follows.

**Dirichilet boundary conditions:**

\[ u(0, t) = 0 \]

\[ u(l, t) = 0 \]

\[ u(0, t) = 0 \] \hspace{1cm} \ldots(13)

**Initial conditions (cauchy conditions at \(t = 0\))**

\[ u(x, 0) = \sin \pi x \hspace{1cm} ; \hspace{0.5cm} 0 \leq x \leq 1 \]
We solve this problem with spline explicit - implicit method and compare the results with Finite Difference Explicit - Implicit and Finite element method as well as exact solution with same boundary and initial condition.

5. Result

**Figure (6.1)**
Comparison Of Error Analysis For Spline Explicit And Finite Difference Explicit (at t = 0.04)

**Figure (6.2)**
Comparison Of Error Analysis For Spline Implicit And Finite Difference Implicit (at t = 0.04)

**Figure (6.4)**
Comparison Of Error Analysis For Spline Explicit- Implicit (at t = 0.04)

**Figure (5.8.2)**
Comparison Of Error Analysis For Spline Explicit-Implicit And Finite Element Results
7 Conclusion:

It is noticed from graph and table that spline solution produces a reduced amount of error than finite difference and finite element method. Analytic solution and spline solution represents a single curve which indicates the accuracy and reliability of spline collocation technique and spline solution are accurate up to five decimal places. Also spline collocation method required compact calculations and besides these two methods, namely explicit and implicit method, implicit methods have more accurate results than the explicit method. This justifies that spline collocation approach to finite difference, finite element approximation as well as analytic solution.

REREFERENCE:

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