A Quintic Spline method for fourth order singularly perturbed Boundary Value Problem of Reaction-Diffusion type

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ABSTRACT

We consider quintic spline collocation for fourth order singularly perturbed boundary value problem in which the highest order derivative is multiplied by a small parameter. The quintic spline collocation method, finite difference method and Spline method due to Blue are tested on an example. The results obtained using quintic spline collocation method are found to be in agreement with the exact solution given in the literature.

KEY WORDS: Quintic spline collocation, Singular perturbation, Fourth order ordinary differential equation, Upper triangular matrix.

1. INTRODUCTION

The approximate solution of boundary value problems with a small parameter affecting highest derivative of the differential equation is described. This class of problems has recently gained importance in the literature for two main reasons. Firstly, they occur frequently in many areas of science and engineering, for example, combustion, chemical reactor theory, nuclear engineering, control theory, elasticity, fluid mechanics, etc. Secondly, the occurrence of sharp boundary-layers as $\varepsilon$, the coefficient of highest derivative, approaches zero creates difficulty for most standard numerical schemes. There exist a variety of techniques for solving singularly perturbed boundary value problems [1-5]. The numerical solution of two point boundary value problems using splines has been considered by many authors [4,6-9] and references therein. Most of the papers concerning computational aspects are confined to second order equations. Only a few results are available for higher order equations. Singularly perturbed higher-order problems are classified on the basis of how the order of the original differential equation is affected if one sets $\varepsilon = 0$ by Roos et al [10], where $\varepsilon$ is a small positive parameter assigned with order of the differential equation. We say that if the order of the differential equation is reduced by one, then the Singular Perturbation Problem (SPP) is said to be of convection diffusion type and if the order is reduced by two, then it is called reaction diffusion type. There are a variety of techniques for solving singularly perturbed boundary value problems. O’Malley [11], Howes [12], and Zhao [13] derived analytical results of third order nonlinear SPPs. Feckan [14] considered higher order problems and his approach is based on the nonlinear analysis involving fixed-point theory, Leray-Schauder theory, etc. A FEM for convection-reaction type problems is described by G. Sun et.al. [15,16] and Semper [17]. Roberts [18] suggested a numerical method of finding an approximate solution for third order ODEs. He introduced a technique called Boundary Value Technique (BVT). In section 2, we suggest spline collocation method to obtain solutions of the Boundary Value Problems (BVPs) of higher order SPODEs. The boundary value technique to find numerical solution for third order SPODEs subject to certain types of boundary conditions is suggested by Valarmathi et al [19]. This work is an extension from third order to fourth order. In the present paper, we solve the fourth order singularly perturbed boundary value problem using quintic spline collocation method.

2. SINGULARLY PERTURBED FOURTH ORDER ORDINARY DIFFERENTIAL EQUATION OF REACTION DIFFUSION TYPE

Consider the boundary value Problem

\[ -\varepsilon y''''(x) + b(x) y'''(x) - c(x) y''(x) = -f(x), \quad x \in D, \]
\[ y(0) = p, \quad y'(0) = q, \quad y''(0) = -r, \quad y'''(1) = s \]

where $\varepsilon > 0$ is a small positive parameter, and $b(x)$, $c(x)$ and $f(x)$ are sufficiently smooth functions satisfying the following conditions:
\[ b(x) \geq \beta > 0, \]
\[ 0 \geq c(x) \geq -\gamma, \quad \gamma > 0, \]
\[ \beta - \theta \gamma > \eta > 0, \quad \theta > 1 \] is arbitrary close to 1, for some \( \eta \)
\[ D = (0, 1), \quad \overline{D} = [0, 1], \]
and \( y \in c^4(D) \cap c^2(\overline{D}) \)

The above problem comes from the reference [20], which states that the deflection of an elastic beam with small flexural rigidity under tension subject to a specified load and stability problems in fluid dynamics leads to the Orr-Sommerfield equation presented by O'Malley[11] and B.Semper[17]. This yields fourth-order singularly perturbed differential equations, whose order decreases by two when one set \( \varepsilon = 0 \). Section 3 describes the quintic spline collocation method to solve such type of singularly perturbed fourth-order ODE of reaction-diffusion type.

### 3 Quintic Spline Collocation Method:

The fifth degree spline is used to find numerical solutions to the boundary value problems discussed in equation (1) together with equation (2). A detailed description of spline functions generated by subdivision is given by De Boor [21].

Consider equally spaced knots of a partition \( \pi \):

\[ a = x_0 < x_1 < x_2 < \ldots < x_n = b \] on \( [a,b] \). Let \( S_{4}[\pi] \) be the space of continuously differentiable, piecewise, Quartic polynomials on \( \pi \). That is, \( S_{4}[\pi] \) is the space of Quartic polynomials on \( \pi \). The Quintic spline is given by Bickley [8] and by Micula et al [22].

\[
s(x) = a_0 + b_0 (x - x_0) + \frac{1}{2} c_0 (x - x_0)^2 + \frac{1}{6} d_0 (x - x_0)^3 + \frac{1}{24} e_0 (x - x_0)^4 + \frac{1}{120} \sum_{k=0}^{n-1} F_k (x - x_k)^5 \tag{3}
\]

where the power function \( (x - x_k)_+ = \max(x - x_k, 0) \), if \( x > x_k \)
\( = 0 \), if \( x \leq x_k \)

Consider a third order boundary value problem of the form

\[
y''(x) + p(x) y''(x) + q(x) y''(x) + r(x) y'(x) + t(x) y(x) = m(x); a \leq x \leq b
\tag{4}
\]

subject to the boundary conditions

\[ \alpha_0 y_0 + \beta_0 y_n + \gamma_0 y''(b) + \delta_0 y'''' = \eta_0 \]
\[ \alpha_1 y_0' + \beta_1 y_n' + \gamma_1 y''(b) + \delta_1 y'''' = \eta_1 \]
\[ \alpha_2 y_0'' + \beta_2 y_n'' + \gamma_2 y''(b) + \delta_2 y'''' = \eta_2 \]

\[ \alpha_3 y_0''' + \beta_3 y_n''' + \gamma_3 y''''(b) + \delta_3 y''''' = \eta_3 \]

where \( y(x), p(x), q(x), r(x), t(x), m(x) \) are continuous functions defined in the interval \( x \in [a,b] \); \( \eta_0, \eta_1, \eta_2, \eta_3 \) are finite real constants.

Let equation (3) be an approximate solution of equation (4), where \( a_0, b_0, c_0, d_0, e_0, F_0, F_1, \ldots, F_{n+1} \) are real coefficients to be determined.

Let \( x_0, x_1, \ldots, x_n \) be \( n+1 \) grid points in the interval \( [a, b] \), so that

\[ x_i = a + ih, \quad i = 0, 1, \ldots, n; \quad x_0 = a, x_n = b, \]

\[ h = (b-a)/n \].

It is required that the approximate solution (3) satisfies the differential equation at the point \( x=x_i \).

Putting (3) with its successive derivatives in (1), we obtain the collocation equations as follows:

\[
\sum_{k=0}^{n-1} e_k \left( x_i - x_k \right)_+ + \frac{1}{2} p(x_i) \left( x_i - x_k \right)_+^2 + \frac{1}{6} q(x_i) \left( x_i - x_k \right)_+^3 + \frac{1}{24} r(x_i) \left( x_i - x_k \right)_+^4 + \frac{1}{120} \sum_{k=0}^{n-1} F_k \left( x_i - x_k \right)_+^5 \right) = m(x_i), \]

\[ i = 0, 1, 2, \ldots, n \tag{6} \]

From boundary conditions,

\[
\sum_{k=0}^{n-1} F_k \left( \frac{\delta_0}{2} (b - x_k)_+^2 + \frac{\gamma_0}{6} (b - x_k)_+^3 \right) + e_0 \left( \frac{\delta_0}{2} (b - a)_+^2 + \frac{\gamma_0}{2} (b - a)_+^2 \right) + \]

\[ d_0 (\delta_0 + \gamma_0 (b - a)) + c_0 (\gamma_0) + b_0 (\beta_0) \]

\[ + a_0 (\alpha_0) = \eta_0 \tag{7} \]
Using the power function \((x - x_0)\), in the above equations a system of \(n+4\) linear equations in \(n+4\) unknowns \(a_0, b_0, c_0, d_0, e_0, e_1, \ldots, e_{n-1}\) is thus obtained. This system can be written in matrix-vector form as follows

\[
AX = B \tag{11}
\]

Where

\[
X = [e_{n-1}, e_{n-2}, \ldots, e_2, e_1, e_0, d_0, c_0, b_0, a_0]^T
\]

\[
B = [\delta_2, \delta_1, \delta_0, m(x_0), m(x_{n-1}), \ldots, m(x_1), m(x_0)]^T
\]

The coefficient matrix \(A\) is an upper triangular Hesseberg matrix with a single lower sub diagonal, principal and upper diagonal having non-zero elements. Because of this nature of matrix \(A\), the determination of the required quantities becomes simple and consumes less time. The values of these constants ultimately yield the Quintic spline \(s(x)\) in equation (3).

In case of nonlinear boundary value problem, the equations can be converted into linear form by quasilinearization method [23] and hence this method can be used as iterative method. The procedure to obtain a spline approximation of \(y_i, (i=0, 1, 2, \ldots j; \text{ where } j \text{ denotes the number of iteration})\) by an iterative method starts with fitting a curve satisfying the end conditions and this curve is designated as \(y_i\). We obtain the successive iterations \(y_i, s\) with the help of an algorithm described as above till desired accuracy.

4 NUMERICAL ILLUSTRATION AND DISCUSSION

Consider the boundary value problem

\[-\varepsilon y^{iv}(x) + 4y^"(x) + y(x) = -f(x),\]

\[y(0) = 1, \quad y(1) = 1, \quad y^"(0) = -1, \quad y^"(1) = -1,\]

where \(f(x) = -\frac{x(1-x)}{8} - \frac{5\varepsilon}{16}\)

\[
5\varepsilon \frac{e^{-2x/\sqrt{\varepsilon}} - e^{-2(l+1)/\sqrt{\varepsilon}} + e^{-2(l-1)/\sqrt{\varepsilon}} - e^{-2(2-l)/\sqrt{\varepsilon}}}{16} e^{-4l/\sqrt{\varepsilon}}
\]

The solutions of above problem are obtained through the Quintic spline collocation method, Blue’s method and Finite difference method which are exhibited in table-1. Blue’s method and Finite difference method give good results, are reflected in table-1. Exact solutions, which are presented by S.
Shanthi et al. [20] are also shown in table-1 and also in figure-1 so that one can easily find the closed form solution of the above problems.

**Table 1  Numerical Solutions of Fourth Order SPODE (h = 0.1, ε = 0.01)**

<table>
<thead>
<tr>
<th>x</th>
<th>Quintic Spline Method</th>
<th>Blue’s Method</th>
<th>Finite Difference Method</th>
<th>Exact Solution</th>
</tr>
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<td>1.0000000</td>
<td>1.0000000</td>
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<td>1.017893</td>
<td>1.010715</td>
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<td>1.0604456</td>
<td>1.0217952</td>
<td>1.018224</td>
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</tbody>
</table>

![Figure-1: Numerical Solutions of Fourth Order SPODE](image)

As mentioned in section 1, the second order singularly perturbed differential equations have been studied extensively, but only few results on the higher-order problems are available in the literature. We presented numerical methods, like spline collocation method, finite difference method, Blue’s method to solve boundary value problems for fourth-order SPODE.

In section 4, the singularly perturbed fourth order ordinary differential equation of reaction diffusion problem is solved using spline collocation method. The comparison table-1 shows that Quintic spline gives better approximation than Blue’s method & Finite difference method and the convergence is also faster. In addition, spline method is easy to implement for boundary value problems for higher order singularly perturbed ordinary differential equations. Figure-1 provide a good agreement of results presenting actual as well as approximate solutions, which proves the reliability of the spline collocation methods.

**REFERENCES**


